Kerr black hole quasinormal frequencies

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Black-hole quasinormal modes (QNM) have been the subject of much recent attention, with the hope that these oscillation frequencies may shed some light on the elusive theory of quantum gravity. We compare numerical results for the QNM spectrum of the (rotating) Kerr black hole with an *exact* formula $\mathrm{Re}\,\omega \to T_{BH}\ln 3 + \Omega m$, which is based on Bohr's correspondence principle. We find a close agreement between the two. Possible implications of this result to the area spectrum of quantum black holes are discussed.

Gravitational waves emitted by a perturbed black hole are dominated by 'quasinormal ringing', damped oscillations with a *discrete* spectrum [1]. At late times, all perturbations are radiated away in a manner reminiscent of the last pure dying tones of a ringing bell [2–5]. The quasinormal mode frequencies (ringing frequencies) are the characteristic 'sound' of the black hole itself, depending on its parameters (mass, charge, and angular momentum).

The free oscillations of a black hole are governed by the well-known Regge-Wheeler equation [6] in the case of a Schwarzschild black hole, and by the Teukolsky equation [7] for the (rotating) Kerr black hole. The black hole QNM correspond to solutions of the wave equations with the physical boundary conditions of purely outgoing waves at spatial infinity and purely ingoing waves crossing the event horizon [8]. Such boundary conditions single out *discrete* solutions ω (assuming a time dependence of the form $e^{i\omega t}$).

The ringing frequencies are located in the complex frequency plane characterized by $\mathrm{Im}\,\omega > 0$. It turns out that for a given angular harmonic index l there exist an infinite number of quasinormal modes, for $n = 0, 1, 2, \ldots$, characterizing oscillations with decreasing relaxation times (increasing imaginary part) [9,10]. On the other hand, the real part of the frequencies approaches an asymptotic *constant* value.

The QNM frequencies, being a signature of the black-hole spacetime are of great importance from the astrophysical point of view. They allow a direct way of identifying the spacetime parameters (especially, the mass and angular momentum of the central black hole). This has motivated a flurry of activity with the aim of computing the spectrum of oscillations (see e.g., [1] for a detailed review).

Recently, the quasinormal frequencies of black holes have acquired a different importance [11–17] in the context of Loop Quantum Gravity, a viable approach to the quantization of General Relativity (see e.g., [18,19] and references therein). These recent studies are motivated by an earlier work of Hod [20]. Few years ago I proposed to use *Bohr's correspondence principle* in order to determine the value of the fundamental area unit in a quantum theory of gravity.

To understand the original argument it is useful to recall that in the early development of quantum mechanics, Bohr suggested a correspondence between classical and quantum properties of the Hydrogen atom, namely that "transition frequencies at large quantum numbers should equal classical oscillation frequencies". The black hole is in many senses the "Hydrogen atom" of General relativity. I therefore suggested [20] a similar usage of the discrete set of black-hole frequencies in order to shed some light on the *quantum* properties of a black hole. However, there is one important difference between the Hydrogen atom and a black hole: while a (classical) atom emits radiation spontaneously according to the (classical) laws of electrodynamics, a *classical* black hole does not emit radiation. This crucial difference hints that one should look for the highly damped black-hole free oscillations [let $\omega = \mathrm{Re}\,\omega + i\mathrm{Im}\,\omega$, then $\tau \equiv (\mathrm{Im}\,\omega)^{-1}$ is the effective relaxation time for the black hole to return to a quiescent state after emitting gravitational radiation. Hence, the relaxation time $\tau \to 0$ as $\mathrm{Im}\,\omega \to \infty$, implying no radiation emission, as should be the case for a classical black hole].

Leaver [9] was the first to address to problem of computing the black hole highly damped ringing frequencies. Nollert [21] found numerically that the asymptotic behavior of the ringing frequencies of a Schwarzschild black hole is given by (we normalize $G = c = 2M = 1$)

$$\omega_n = 0.0874247 + \frac{i}{2}\left(n + \frac{1}{2}\right) , \qquad (1)$$

as $n \to \infty$ [22]. The asymptotic behavior Eq. (1) was later verified by Andersson [23] using an independent analysis. In Ref. [20] I realized that the asymptotic real part of the frequencies equals $\ln 3/8\pi$, and proposed a heuristic picture (based on thermodynamic and statistical physics arguments) trying to explain this fact. Most recently, Motl [13] has given an analytical proof for this conjecture.

Using the relation $A = 16\pi M^2$ for the surface area of a Schwarzschild black hole, and $\Delta M = E = \hbar\omega$ one finds $\Delta A = 4\ell_P^2 \ln 3$ with the emission/absorption of a quantum, where ℓ_P is the Planck length. Thus, we concluded

that the area spectrum of the quantum Schwarzschild black hole is given by

$$A_n = 4\ell_P^2 \ln 3 \cdot n \quad ; \quad n = 1, 2, \dots \quad . \tag{2}$$

This result is remarkable from a statistical physics point of view. It does not relay in any way on the well known thermodynamic relation between black-hole surface area and entropy $S_{BH} = \frac{1}{4}A$ [24]. In the spirit of Boltzmann-Einstein formula in statistical physics, Mukhanov and Bekenstein [25–27] relate $g_n \equiv exp[S_{BH}(n)]$ to the number of microstates of the black hole that correspond to a particular external macrostate. In other words, g_n is the degeneracy of the nth area eigenvalue. The accepted thermodynamic relation between black-hole surface area and entropy [24], combined with the requirement that g_n has to be an integer for every n, actually enforce a factor of the form $4 \ln k$ (with $k = 2, 3, \dots$) in Eq. (2). We have shown that the value $k = 3$ is the only one compatible both with the area-entropy thermodynamic relation for black hole, and with Bohr's correspondence principle as well.

Bekenstein [24,27,28] (see also [20,29] and references therein) has given evidence for the existence of a universal (i.e., independent of the black-hole parameters: mass, charge, and angular momentum) area spacing for quantum black holes. This, combined with the universality of the black-hole entropy (i.e., its direct thermodynamic relation to the black-hole surface area) suggest that the area spectrum Eq. (2) should be valid for rotating black holes as well. In fact, our analysis leads to a natural conjecture for the asymptotic behavior of the highly damped quasinormal frequencies of a generic (rotating) Kerr black hole. First, we use the first law of black-hole thermodynamics

$$\Delta M = T_{BH} \Delta S + \Omega \Delta J , \tag{3}$$

where $T_{BH} = (r_+ - r_-)/A$ is the Bekenstein-Hawking temperature, and $\Omega = 4\pi a/A$ is the angular velocity of the black-hole horizon [$r_\pm = M \pm (M^2 - a^2)^{1/2}$ are the black hole (event and inner) horizons, and $a = J/M$ is the black hole angular momentum per unit mass]. Taking cognizance of Eqs. (2) and (3) [together with the relation $S = \frac{1}{4}A$], one finds

$$Re\omega \to T_{BH} \ln 3 + \Omega m , \tag{4}$$

where m is the azimuthal eigenvalue of the oscillation. The corresponding problem of the Hydrogen atom in quantum mechanics hints that the formula should be valid in the $l = m$ case.

It should be emphasized that the asymptotic behavior of the black hole ringing frequencies is known only for the simplest case of a Schwarzschild black hole. Less is known about the corresponding QNM spectrum of the (rotating) Kerr black hole [9,30,31]. This is a direct consequence of the numerical complexity of the problem. Onozawa [31] computed the first nine frequencies of the Kerr black hole. It is of great interest to compare the conjectured asymptotic behavior given by Eq. (4) with the results of direct numerical computations.

Figure 1 displays $Re\omega$ for the Kerr black hole, as computed numerically in [31], and compare it with the analytically conjectured formula Eq. (4) [32]. We find that the predicted results agree with the numerically computed ones to within $\sim 5\%$.
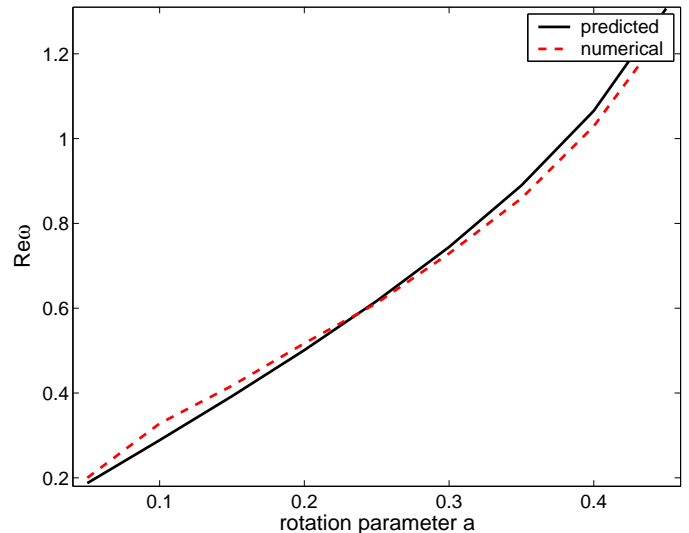

FIG. 1. Real part of the Kerr black hole QNM frequencies as a function of the black hole rotation parameter a. The numerical results are for gravitational quasinormal modes with $l = m = 2$. The *predicted* values (solid line) agree with the *numerically* computed ones (dashed line) to within $\sim 5\%$.

Note that the imaginary parts of the asymptotic QNM frequencies are *equally* spaced in the Schwarzschild case [see Eq. (1)], with a spacing of $1/4M = 2\pi T_{BH}$. In order to check if this relation holds true for (generic) Kerr black holes as well, we display in Fig. 2 the spacing $\Delta(Im\omega)$ using the numerical data of [31], and compare it with a predicted value of $2\pi T_{BH}$. For $a > 0.1$, we find that the numerically computed values agree with the predicted ones to within $\sim 7\%$.

In summary, based on Bohr's correspondence principle we have conjectured a simple formula for the asymptotic QNM frequencies of a generic (rotating) Kerr black hole. We find a good agreement between the theoretically predicted frequencies and the numerically computed ones. This agreement lands support for the validity of the area spectrum Eq. (2).
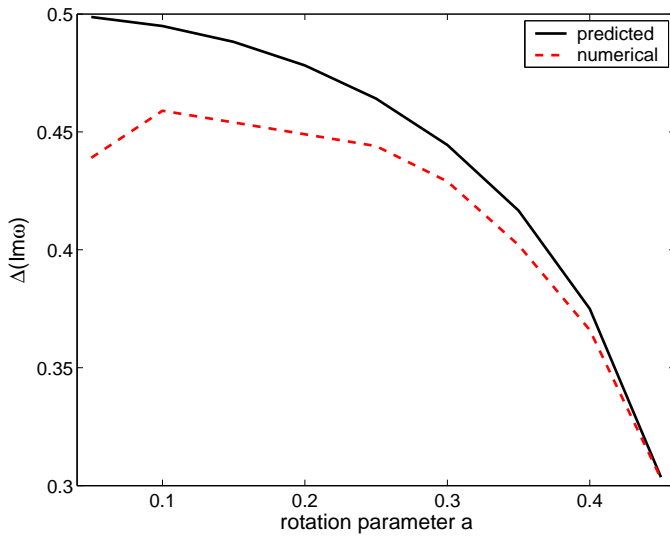
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FIG. 2. Spacing of $\text{Im}\omega$ as a function of the black hole rotation parameter a. For $a > 0.1$, the *predicted* values (solid line) agree with the *numerically* computed ones (dashed line) to within $\sim 7\%$.

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